MATH 773: Assignment 5

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Problem 1

Proof We have

$$\psi(\theta) = a\theta + \frac{1}{2}\sum_{j=1}^{m}\left(\frac{\theta^2 b_j^2}{1-2\theta b_j} - \ln\left(1-2\theta\lambda_j\right)\right)$$

$$Q = a + \sum_{j=1}^{m} b_j Z_k + \sum_{j=1}^{m}\lambda_j Z_j^2$$

The CDF of $Z \sim N(0, I)$ under new measure $\mathbb{P}_\theta$ is

$$\begin{aligned}
\mathbb{P}_\theta(Z_1 \le z_1, ..., Z_m \le z_m) &= \mathbb{E}\left(e^{\theta Q - \psi(\theta)}\mathbb{1}\left(Z_1 \le z_1, ..., Z_m \le z_m\right)\right)\\
&= \int\cdots\int_{Z_j=-\infty}^{z_j} e^{\theta Q-\psi(\theta)}\frac{e^{-\frac{1}{2}\vec{Z}'\Sigma^{-1}\vec{Z}}}{(2\pi)^{\frac{m}{2}}|\Sigma|^{\frac{1}{2}}}\mathrm{d}Z_j\\
&= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\cdots\int_{Z_j=-\infty}^{z_j} e^{\theta Q-\psi(\theta)-\frac{1}{2}\sum_{j=1}^m Z_j^2}\mathrm{d}Z_j\\
&= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\cdots\int_{Z_j=-\infty}^{z_j} e^{\theta\left(a+\sum_{j=1}^m b_j Z_k+\sum_{j=1}^m \lambda_j Z_j^2\right)-a\theta+\frac{1}{2}\sum_{j=1}^m\left(\frac{\theta^2 b_j^2}{1-2\theta b_j}-\ln(1-2\theta\lambda_j)\right)-\frac{1}{2}\sum_{j=1}^m Z_j^2}\mathrm{d}Z_j\\
&= \frac{1}{(2\pi)^{\frac{m}{2}}}\int\cdots\int_{Z_j=-\infty}^{z_j} e^{\frac{1}{2}\sum_{j=1}^m \ln(1-2\theta\lambda_j)}e^{\sum_{j=1}^m\left(\theta\lambda_j-\frac{1}{2}\right)Z_j^2}e^{\sum_{j=1}^m \theta\lambda_j Z_j-\frac{1}{2}\sum_{j=1}^m\frac{\theta^2 b_j^2}{1-2\theta b_j}}\\
&= \frac{\left(\prod_{j=1}^m\left(1-2\theta\lambda_j\right)\right)^{\frac{1}{2}}}{(2\pi)^{\frac{m}{2}}}\int\cdots\int_{Z_j=-\infty}^{z_j} e^{\sum_{j=1}^m\left(\theta\lambda_j-\frac{1}{2}\right)Z_j^2+\sum_{j=1}^m\theta\lambda_j Z_j-\frac{1}{2}\sum_{j=1}^m\frac{\theta^2 b_j^2}{1-2\theta b_j}}
\end{aligned}$$

We want this to be the CDF of Z under measure $\mathbb{P}_\theta$, i.e. $Z \sim N\left(\vec{\mu}, \Sigma'\right)$ where

$$\mu_j\left(\theta\right) = \frac{\theta b_j}{1-2\theta\lambda_j},$$

and Σ' is a diagonal matrix with diagonal entries

$$\sigma_j^2(\theta) = \frac{1}{1-2\theta\lambda_j}.$$

Then the CDF of $Z \sim N\left(\vec{\mu}, \Sigma'\right)$ is

$$\begin{aligned}
\frac{e^{-\frac{1}{2}(\vec{z}-\vec{\mu})'\Sigma'^{-1}(\vec{z}-\vec{\mu})}}{(2\pi)^{\frac{m}{2}}|\Sigma'|^{\frac{1}{2}}} &= \frac{1}{(2\pi)^{\frac{m}{2}}\left(\prod_{j=1}^m\left(\frac{1}{1-2\theta\lambda_j}\right)\right)^{\frac{1}{2}}}e^{-\frac{1}{2}\sum_{j=1}^m\left(z_j-\frac{\theta b_j}{1-2\theta\lambda_j}\right)(1-2\theta\lambda_j)\left(z_j-\frac{\theta b_j}{1-2\theta\lambda_j}\right)}\\
&= \frac{\left(\prod_{j=1}^m\left(1-2\theta\lambda_j\right)\right)^{\frac{1}{2}}}{(2\pi)^{\frac{m}{2}}}e^{-\frac{1}{2}\sum_{j=1}^m\left((1-2\theta\lambda_j)z_j^2-2\theta b_j z_j+\frac{\theta^2 b_j^2}{1-2\theta b_j}\right)}\\
&= \frac{\mathrm{d}}{\mathrm{d}\vec{z}}\mathbb{P}_\theta(Z_1 \le z_1, ..., Z_m \le z_m)
\end{aligned}$$

Therefore under the new measure $\mathbb{P}_\theta$ Z is indeed still a multivariate normal random variable, with mean $\vec{\mu}$ and variance matrix Σ'.

□

Problem 2

Proof From the text we know that

$$\mathbb{E}\left(e^{\theta Q_x}\middle| Y\right) = e^{\alpha(\theta)Y}\prod_{j=1}^{m}\frac{1}{\sqrt{1-2\theta\lambda_j}}$$

where

$$\alpha(\theta) = (a-x)\frac{\theta}{\nu} + \frac{1}{2}\sum_{j=1}^{m}\frac{\theta^2 b_j^2 \nu}{1-2\theta\lambda_j}$$

We also know that

$$\phi_x(\theta) = \mathbb{E}\left(e^{\theta Q_x}\right) = \phi_Y(\alpha(\theta))\prod_{j=1}^{m}\frac{1}{\sqrt{1-2\theta\lambda_j}}$$

and its cumulant generating function

$$\psi_x(\theta) = \ln\phi_x(\theta) = \ln\mathbb{E}\left(e^{\theta Q_x}\right)$$

The density of Y under new measure $\mathbb{P}_\theta$ can be found by differentiating

$$\begin{aligned}
\mathbb{P}_\theta(Y \le y) &= \mathbb{E}\left(\frac{\mathrm{d}\mathbb{P}_\theta}{\mathrm{d}\mathbb{P}}\mathbb{1}(Y \le y)\right) \\
&= \mathbb{E}\left(e^{\theta Q_x - \psi_x(\theta)}\mathbb{1}(Y \le y)\right) \\
&= e^{-\psi_x(\theta)}\mathbb{E}\left(\mathbb{E}\left(e^{\theta Q_x}\mathbb{1}(Y \le y)\middle| Y\right)\right) \\
&= e^{-\psi_x(\theta)}\mathbb{E}\left(e^{\alpha(\theta)Y}\prod_{j=1}^{m}\frac{1}{\sqrt{1-2\theta\lambda_j}}\mathbb{1}(Y \le y)\right) \\
&= \frac{1}{\mathbb{E}\left(e^{\theta Q_x}\right)}\int_0^y e^{\alpha(\theta)\zeta}\prod_{j=1}^{m}\frac{1}{\sqrt{1-2\theta\lambda_j}}f_Y(\zeta)\mathrm{d}\zeta \\
&= \frac{1}{\phi_Y(\alpha(\theta))}\int_0^y e^{\alpha(\theta)\zeta}f_Y(\zeta)\mathrm{d}\zeta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow f_{Y,\theta}(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}_\theta(Y \le y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\phi_Y(\alpha(\theta))}\int_0^y e^{\alpha(\theta)\zeta}f_Y(\zeta)\mathrm{d}\zeta\right) \\
&= \frac{e^{\alpha(\theta)y}}{\phi_Y(\alpha(\theta))}f_Y(y)
\end{aligned}$$

□

Problem 3

Portfolio A: Standard Monte Carlo

```
clc;
clear;
rand('state',0);
%Portfolio A: Short positions in 10 calls and 5 puts on each of 10
%underlying assets, all options expiring in 0.1 years.
%% Initialize portfolio parameters
volatility=0.3;
rate=0.05;
maturity=0.1;
NCall=10;
NPut=5;
NAsset=10;
NRepl=120000;
QEBinN=600; %Bin size used for VaR quantile estimation.
QEIndex=2; %Estimated quantile=1-QEIndex/(NRepl/QEBinSize)=0.99
assetprice0=ones(NAsset,1);
assetprice=zeros(NAsset,NRepl);
```

```
strikes=[[0.55:0.1:1.45], [0.8:0.1:1.2]];
payoff=zeros(1,NRepl);
%% Calculate theoretical portfolio value
TPV=0;
for i=1:NAsset
    call=zeros(1,NCall+NPut);
    put=zeros(1,NCall+NPut);
    for j=1:NCall+NPut
        [call(j),put(j)]=blsprice(assetprice0(i),strikes(j),rate,maturity,volatility,0);
    end
    TPV=TPV+sum([call(1:NCall),put(NCall+1:NCall+NPut)]);
end
%% Start Monte Carlo runs
assetprice=diag(assetprice0)*exp((rate-volatility^2/2)*maturity+sqrt(maturity)*...
    volatility*randn(size(assetprice)));
%% Calculate option prices from simulation results
for j=1:NRepl
    for i=1:NAsset
        payoff(j)=payoff(j)+sum([max(assetprice(i,j)-strikes(1:NCall),0),...
            max(strikes(NCall+1:NCall+NPut)-assetprice(i,j),0)]);
    end
end
%% Calculate simulated portfolio loss
SPL=TPV-payoff;
%% Calculate VaR from quantile estimation
VaRQE=zeros(1,QEBinN);
QEBinSize=NRepl/QEBinN;
for i=1:QEBinSize
    VaRQE(i)=quantile(SPL(1+(i-1)*QEBinSize:i*QEBinSize),0.01);
end
[mean,sigma,CI]=normfit(VaRQE)

mean =

   -0.9222


sigma =

    1.3205


CI =

   -1.0280
   -0.8163
```

▷

Portfolio B: Standard Monte Carlo

```
clc;
clear;
rand('state',0);
%Portfolio B: Same as A, but with the number of puts increase to produce a
```

```
%net delta zero.
%% Initialize portfolio parameters
volatility=0.3;
rate=0.05;
maturity=0.1;
NCall=10;
NPut=5;%Increase number of puts in portfolio
NAsset=10;
NRepl=120000;
QEBinN=600; %Bin size used for VaR quantile estimation.
QEIndex=2; %Estimated quantile=1-QEIndex/(NRepl/QEBinSize)=0.99
assetprice0=ones(NAsset,1);
assetprice=zeros(NAsset,NRepl);
strikes=[[0.55:0.1:1.45], [0.8:0.1:1.2]];
payoff=zeros(1,NRepl);
%% Determine number of puts to be added to obtain 0 net delta
CallDelta=zeros(NAsset,1);
PutDelta=zeros(NAsset,1);
ExcessDelta=zeros(2,1);
for i=1:NAsset
    CallDelta(i)=sum(normcdf((log(assetprice0(i)./strikes(1:NCall))+...
        (rate+volatility^2/2)*maturity)/(volatility*sqrt(maturity))));
    PutDelta(i)=sum(normcdf((log(assetprice0(i)./strikes(NCall+1:NCall+NPut))...
        +(rate+volatility^2/2)*maturity)/(volatility*sqrt(maturity)))-1);
end
ExcessDelta(1)=sum(CallDelta)+sum(PutDelta);
ExcessDelta(2)=ExcessDelta(1);
%ExcessDelta determined. Try to find countering put option number.
n=0;
AddPutStrike=[1-0.1*n:0.1:1+0.1*n];
AddPutDelta=FindPutDelta(assetprice0,AddPutStrike,rate,volatility,maturity);
while (n<10 && AddPutDelta+ExcessDelta(2)>0)
    n=n+0.5;
    AddPutStrike=[1-0.1*(n+0.5):0.1:1+0.1*(n+0.5)];
    AddPutDelta=FindPutDelta(assetprice0,AddPutStrike,rate,volatility,maturity);
end
AddPutStrike=[1-0.1*n:0.1:1+0.1*n];
AddPutDelta=FindPutDelta(assetprice0,AddPutStrike,rate,volatility,maturity);
ExcessDelta(2)=AddPutDelta+ExcessDelta(2);
%Used Maple to solve for a strike price 0.9891375095. MATLAB solving sucks!
K=0.9891375095;
AddPutStrike=[AddPutStrike,K];
AddPutDelta=FindPutDelta(assetprice0,AddPutStrike,rate,volatility,maturity);
ExcessDelta(2)=AddPutDelta+ExcessDelta(1);%Check the net delta is approximately 0
NPut=NPut+length(AddPutStrike);%Update Strike counts and values
strikes=[strikes, AddPutStrike];
%% Calculate theoretical portfolio value
TPV=0;
for i=1:NAsset
    call=zeros(1,NCall+NPut);
    put=zeros(1,NCall+NPut);
    for j=1:NCall+NPut
        [call(j),put(j)]=blsprice(assetprice0(i),strikes(j),rate,maturity,volatility,0);
    end
```

```
    TPV=TPV+sum([call(1:NCall),put(NCall+1:NCall+NPut)]);
end
%% Start Monte Carlo runs
assetprice=diag(assetprice0)*exp((rate-volatility^2/2)*maturity+sqrt(maturity)*...
    volatility*randn(size(assetprice)));
%% Calculate option prices from simulation results
for j=1:NRepl
    for i=1:NAsset
        payoff(j)=payoff(j)+sum([max(assetprice(i,j)-strikes(1:NCall),0),...
            max(strikes(NCall+1:NCall+NPut)-assetprice(i,j),0)]);
    end
end
%% Calculate simulated portfolio loss
SPL=TPV-payoff;
%% Calculate VaR from quantile estimation
VaRQE=zeros(1,QEBinN);
QEBinSize=NRepl/QEBinN;
for i=1:QEBinSize
    VaRQE(i)=quantile(SPL(1+(i-1)*QEBinSize:i*QEBinSize),0.01);
end
[mean,sigma,CI]=normfit(VaRQE)

mean =

   -0.7176


sigma =

    1.0302


CI =

   -0.8002
   -0.6350
```

The FindPutDelta function is coded as follows:

```
function delta=FindPutDelta(assetprice0,m,rate,volatility,maturity)
NAsset=length(assetprice0);
APD=zeros(NAsset,1);
for i=1:NAsset
    APD(i)=sum(normcdf((log(assetprice0(i)./m)+(rate+volatility^2/2)*maturity)/...
        (volatility*sqrt(maturity)))-1);
end
delta=sum(APD);
```

▷

Portfolio C: Standard Monte Carlo
This is basically A, just change the first part of the code:

```
%% Initialize portfolio parameters
volatility=0.3;
rate=0.05;
```

```
maturity=0.1;
NCall=10;
NPut=10;
NAsset=100;%<---change here
NRepl=120000;
QEBinN=600; %Bin size used for VaR quantile estimation.
QEIndex=2; %Estimated quantile=1-QEIndex/(NRepl/QEBinSize)=0.99
assetprice0=ones(NAsset,1);
assetprice=zeros(NAsset,NRepl);
strikes=[[0.55:0.1:1.45], [0.55:0.1:1.45]];%<---change here
payoff=zeros(1,NRepl);
......
...

mean =

   -1.5974


sigma =

    2.2746


CI =

   -1.7798
   -1.4150
```

Summary of simulation:

Portfolio	x	$\hat{\mathbb{P}}^{MC}(L > x)$	$\|99\% CI\|$	$\hat{\mathbb{P}}^{IS}(L > x)$	$\|99\% CI\|$	Reduction Factor
A	-0.9222	1%	0.2117	n/a	n/a	n/a
B	-0.7176	1%	0.1652	n/a	n/a	n/a
C	-1.5974	1%	0.3648	n/a	n/a	n/a

□